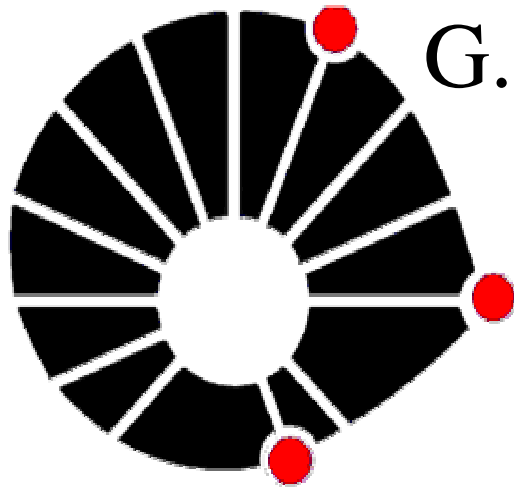




G.Torrieri



UNICAMP

How to make sense of a fluid with
20 particles?

2504.17152 (PRD) with G.M.Sampaio, G.R.Soares

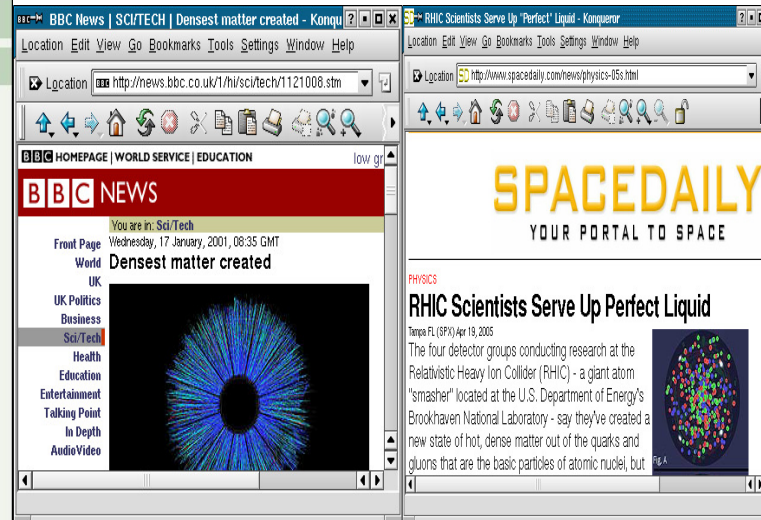
Cover of PRL!!!!

PHYSICAL
REVIEW
LETTERS

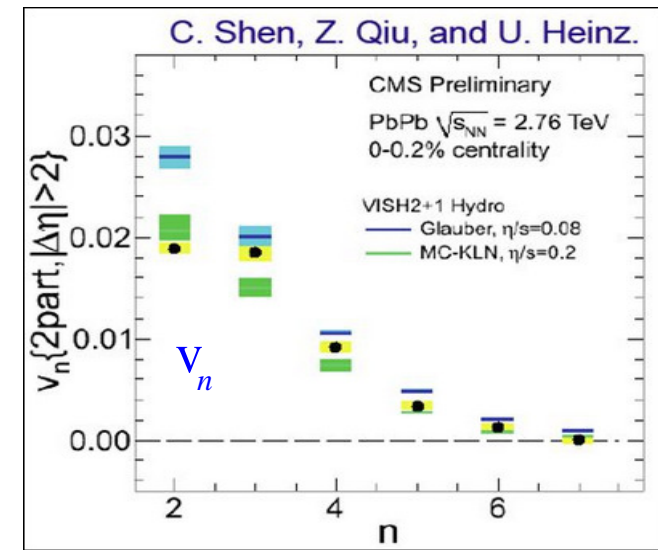
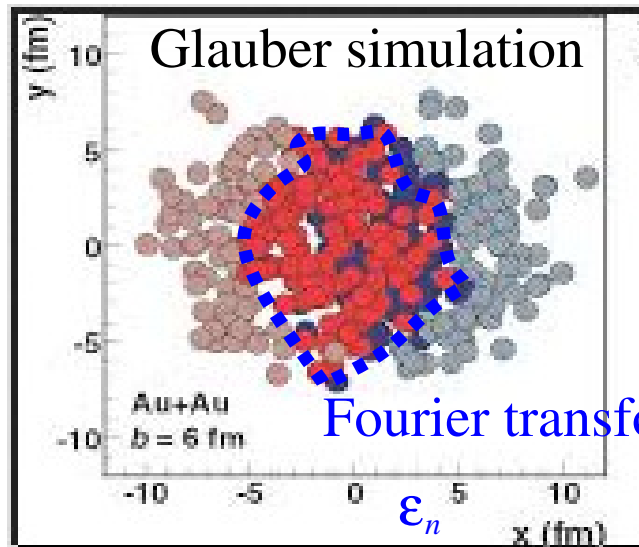


BBC!

SPACE
DAILY!



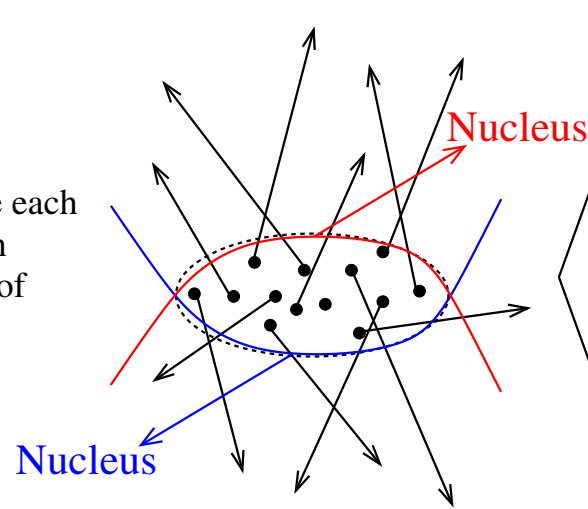
Heavy ion physicists found the perfect liquid! our field redefined by this
Revived interest in relativistic fluid dynamics, developed in 1960-1980s by
general relativists and astrophysicists!



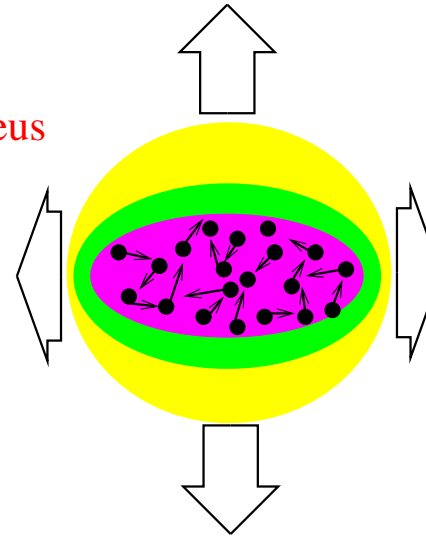
$$\frac{dN}{p_T dp_T dy d\phi} = \frac{dN}{p_T dp_T dy} [1 + 2v_n(p_T, y) \cos(n(\phi - \phi_0(n, p_T, y)))]$$

Given reasonable estimate of initial conditions Fits ideal hydro ,
fitted upper limit on viscosity low Spurned a lot of theoretical and
numerical/phenomenological development of relativistic hydrodynamics.

A "dust"
 Particles ignore each other, their path is independent of initial shape



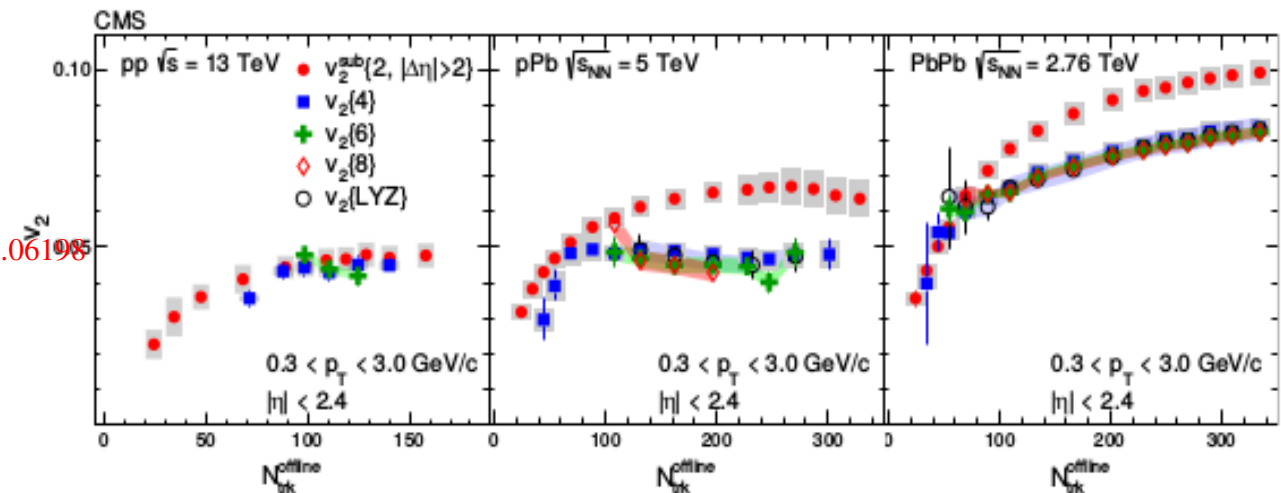
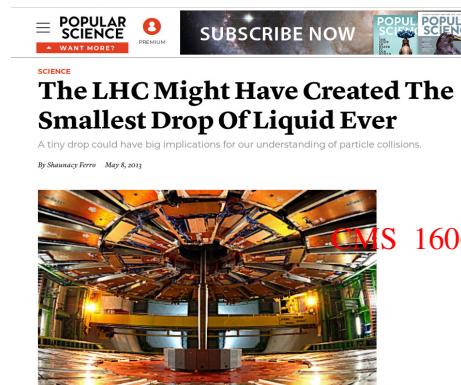
A "fluid"
 Particles continuously interact. Expansion determined by density gradient (shape)



$$\frac{dN}{d^3p} = \frac{dN}{p_T dp_T dy} [1 + 2v_n(p_T, y) \cos(n(\phi - \phi_0))] \quad \begin{aligned} v_n|_{hydro} &= \langle \text{Multipole} \rangle_{volume} \\ v_n|_{MC,exp} &\sim \langle \cos n\phi \rangle \sim c_\infty \end{aligned}$$

Not just local angular momentum or microscopic correlations:
 ZDC/Spectators, correlations survive large rapidity separations "True
 Collectivity" Same v_n appears in $\forall$ n-particle correlations , $\left\langle \frac{dN}{d\phi_1} \frac{dN}{d\phi_2} \dots \right\rangle$

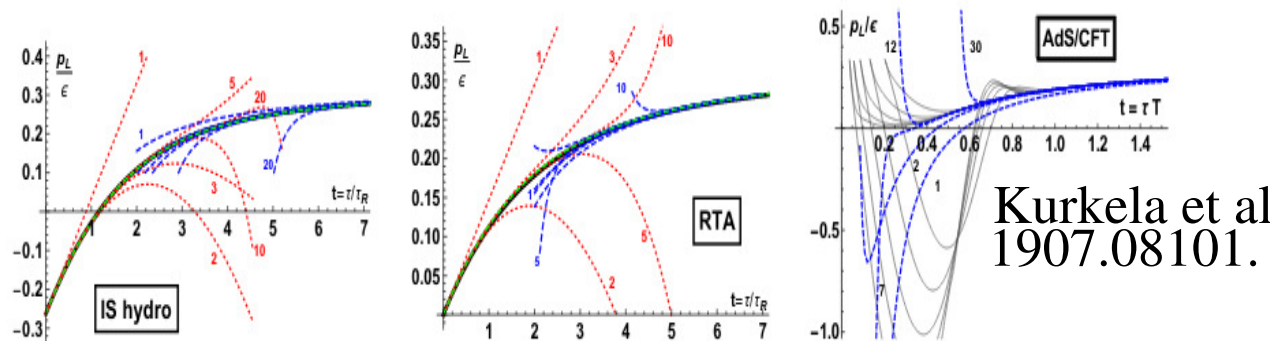
But then LHC switched on and we got a surprise and a conceptual challenge!



1606.06198 (CMS) : When you consider geometry differences and multi-particle cumulants (remove momentum conservation), hydro with $\mathcal{O}(20)$ particles "just as collective" as for 1000. Fluctuations "irrelevant" even when they dominate! 20205: o cumulants ≥ 12 , $N_{ch} \sim 10$

Hydrodynamics in small systems: “hydrodynamization” /” fake equilibrium” ?

A lot more work in both AdS/CFT and transport theory about “hydrodynamization” /” Hydrodynamic attractors”



Fluid-like systems far from equilibrium (**large gradients**)! Usually from 1D solution of Boltzmann and AdS/CFT EoMs! “hydrodynamics converges even at large gradients with no thermal equilibrium”

But the issue is **not big gradients** but **small N_{dof}** ! No Molecular chaos/large N_c , **Ensemble averaging!** , $\langle F(\{x_i\}, t) \rangle \neq F(\{\langle x_i \rangle\}, t)$

Hydrodynamics: an "effective theory" of averages $\langle \dots \rangle$ and thermalization

$$\langle T_{\mu\nu} \rangle = (e + P(e))u_\mu u_\nu + P(e)g_{\mu\nu} + \Pi_{\mu\nu} \quad , \quad \langle J^\mu \rangle = \rho u^\mu + q^\mu$$

At rest w.r.t. u^μ $\langle T_{\mu\nu} \rangle = \text{Diag}(e(p, \mu), p, p, p)$, $\langle J_\mu \rangle = (\rho(p, \mu), \vec{0})$

Makes system solvable just from conservation laws and EoS:

$$\partial_\mu \langle T^{\mu\nu} \rangle = \partial_\mu \langle J^\mu \rangle = 0, p = p(e, \mu), \rho = \rho(e, \mu)$$

Relaxational equation for $\Pi_{\mu\nu}$ (entropy $Ts = \Pi_{\mu\nu} \partial^\mu \beta^\nu$,and $\partial_\mu (su^\mu) \geq 0$)

$$\mathcal{D}(\tau_\pi, T) \Pi_{\mu\nu} + \Pi_{\mu\nu} + \mathcal{O}(\mathcal{D}^{n \geq 2}, \Pi^{n \geq 2}, T^{n \geq 2}, u^{n \geq 2}) = \eta \times \mathcal{O}(\partial u, \partial T) + \mathcal{O}(\partial^{n > 2})$$

(Navier Stokes $\tau_\pi \rightarrow 0$ **acausal** $\Pi_{\mu\nu}$ as a DoF "regularization")

A series whose "small parameter" $K \sim \frac{l_{\text{micro}}}{l_{\text{macro}}} \sim \frac{\eta}{sT} \nabla u \sim \tau_\Pi \nabla u$ and the transport coefficients calculable from asymptotic correlators (Kubo)

Non-relativistic version still considered beautiful and profound, but with relativity... Issues with causality and diffusion give complications

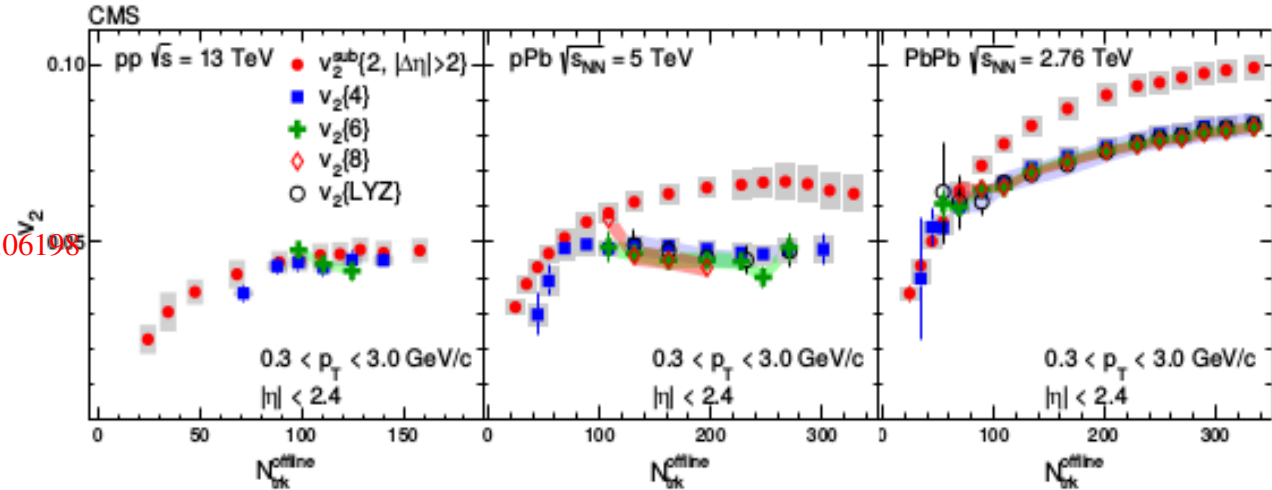
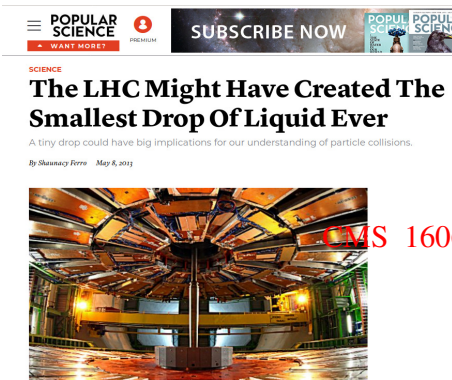
u_μ **ambiguous** many definitions: Landau $u_\mu \propto s_\mu$ Eckart $u_\mu \propto J_\mu$ BDNK: No relaxational DoF, $\Pi_{\mu\nu} \propto \partial u$ Price: complicated anisotropic $u_\mu(T_{\mu\nu})$ We think flow is "clear", so this is a bit strange . choices supposed to be field redefinitions but give slightly different dynamics Geroch, Lindblom,...: when "corrections small" all theories good, when large none good. But no rigorous understanding of this!

$\Pi_{\mu\nu}$ **ambiguous** can even be eliminated as a DOF ($\sim \partial u$ by carefully choosing u_μ (BDNK)) Is it a physical quantity? An observable?

Entropy is ambiguous it's definition depends on the definitions above. Yet from statistical mechanics , as long as microstates are local, it should not be ambiguous! of course entropy related to fluctuations

Fluctuations... $\langle (\Delta T_{\mu\nu})^2 \rangle$ Is not the same as $\langle T_{\mu\nu} \rangle - \langle T_{\mu\nu} \rangle_{eq}$

- One can define linearly, with a Langevin-like fluctuation-dissipation relation **but contradicts experiment!**



- Exact theory strongly depends on u_μ convention! Also on pseudogauge! **but if field redefinition, does "everything" fluctuate?**
What if fluctuation of $u_\mu, T, \Pi_{\mu\nu}$ leave $T_{\mu\nu}$ invariant?

More concretely

A theorist will say that fluctuations of e.g. $\delta\Pi_{\mu\nu}, \delta f(x, p)$ produce "non-hydrodynamic modes", "stochastic transport coefficients", "long-time tails", ... , sensitive to underlying theories, and hydrodynamics is easy to break down to a non-universal dynamics.

An experimentalist measures neither $\Pi_{\mu\nu}$ nor f but rather, e.g.

$$\frac{dN}{dy p_T dp_T d\phi} \equiv \frac{dN}{dy p_T dp_T} [1 + 2v_n(p_T, y) \cos(n(\phi - \phi_{0n}))]$$

i.e. gradients of $T_{\mu\nu}$, entropy : $v_n \equiv \langle \cos(n(\phi - \phi_0)) \rangle$

Most theorists treat it as an average, but This is a cumulant of $\mathcal{O}(\infty)$ so sensitive to non-hydrodynamic modes. Yet experiment finds hydro everywhere they look! Can your non-hydro mode be my fluctuating sound-wave? Can we tell, in principle?

Deriving Hydro $\lim_{k \rightarrow 0} \frac{1}{k} \text{Im} \int d^3x e^{ikx} [T_{xy}(0), T_{xy}(x)] \rightarrow \eta \nabla u \ll Ts$

QFT transport coefficients plagued by divergences, need truncation
Schwinger-Keldysh/Kubo asymptotic correlators “do not commute” with soft thermal fluctuations Moore/Ghiglieri/...

Boltzmann equation Sequential scattering and Molecular chaos. 1st term in series (convergent?), Lose microscopic correlations

AdS/CFT strong coupling and large N_c , also lose microscopic correlations

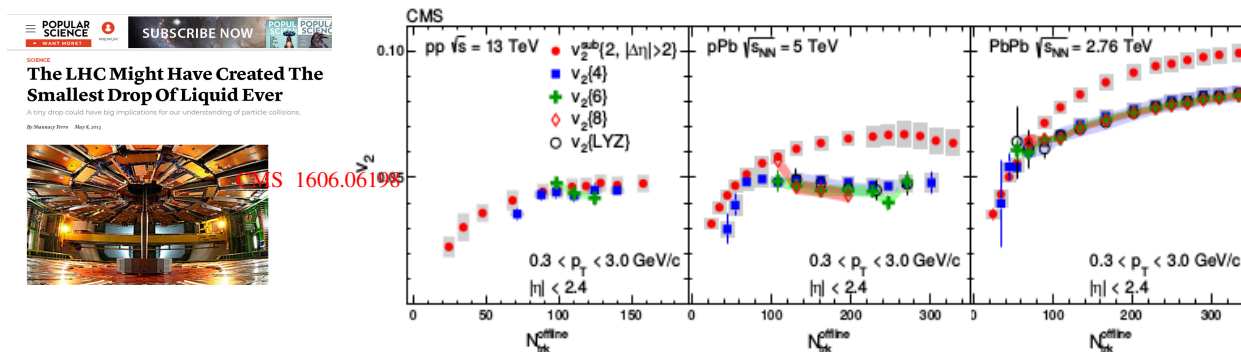
Molecular dynamics keeps microscopic correlations, lose Lorentz invariance (in practice not a problem unless number-changing processes)

Basic problem with correlations/fluctuations on scale of gradients!
Ambiguity in flow, $\Pi_{\mu\nu}$ comes from here!

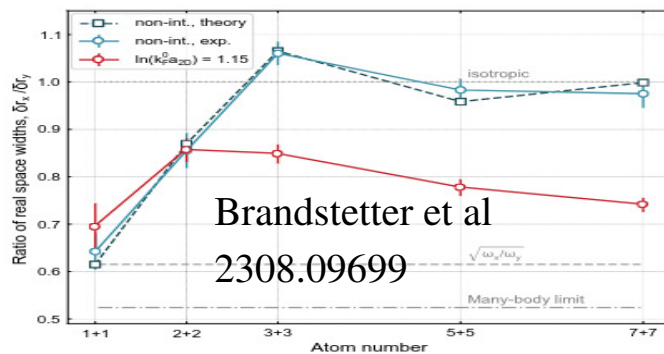
In brief most microscopic approaches to EFT hydrodynamics assume that

$$l_{micro} \ll l_{mfp} \ll L_{macro} \quad , \quad l_{mfp} \sim \eta/(sT) \quad , \quad l_{micro} \sim n_{dof}^{-1/3}$$

Most work around **second inequality** , but **first inequality** crucial for theories behind hydro Boltzmann defined in the **Grad limit** ($n_{dof}^{2/3} \rightarrow \infty, \sigma \rightarrow 0$), **AdS/CFT** the **'t Hooft limit** $N_c^2 \rightarrow \infty, g_{YM} \rightarrow 0$ But both falsified by small systems hydro! **Most attractor/hydrodynamization/... calculations affected!** Issue is not gradients but N_{dof} breaking fluctuation/dissipation!



Not just in heavy ions



The
Brazil
nut effect



Empirically, strongly coupled systems with enough thermal energy seem to be "fluid" even with a small number of DoFs. EFT does not explain this! The role of fluctuations in hydrodynamics, and of the exact relation of statistical physics and hydrodynamics, are still ambiguous and this is related to experimental puzzles How many DoFs make a fluid?

What all this converges to...

Goodstein
"States
of
matter"

THERMODYNAMICS AND STATISTICAL MECHANICS

1.1 INTRODUCTION: THERMODYNAMICS AND STATISTICAL MECHANICS OF THE PERFECT GAS

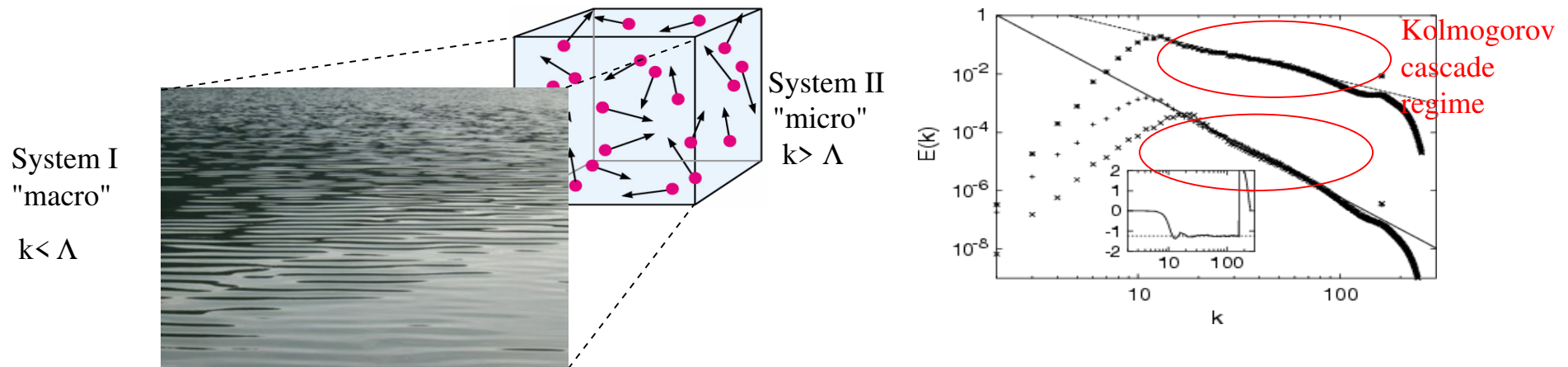
Ludwig Boltzmann, who spent much of his life studying statistical mechanics, died in 1906, by his own hand. Paul Ehrenfest, carrying on his work, died similarly in 1933. Now it is our turn to study statistical mechanics.

Perhaps it will be wise to approach the subject cautiously. We will begin

*Danger, foundations of
stat.mech. ahead!
(Poncaire, Khinchin,...)*

Is that while hydrodynamics is based on statistical mechanics the two do not work together at a deep level!

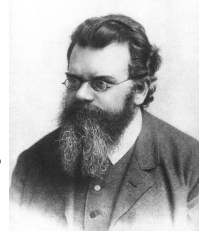
A final issue: Entropy current not clearly connected to energy-momentum current, need microscopic theory to "select good EFT" (2nd law)



At best related to **stability** (sound waves don't explode) and **causality** (sound waves $dw/dk \leq c$) What happens when "mesoscopic DoFs" (turbulence) and microscopic Dof of similar order? Stat mech related to chaos



the battle
of the entropies



and the
unreasonable
effectiveness
of stat mech



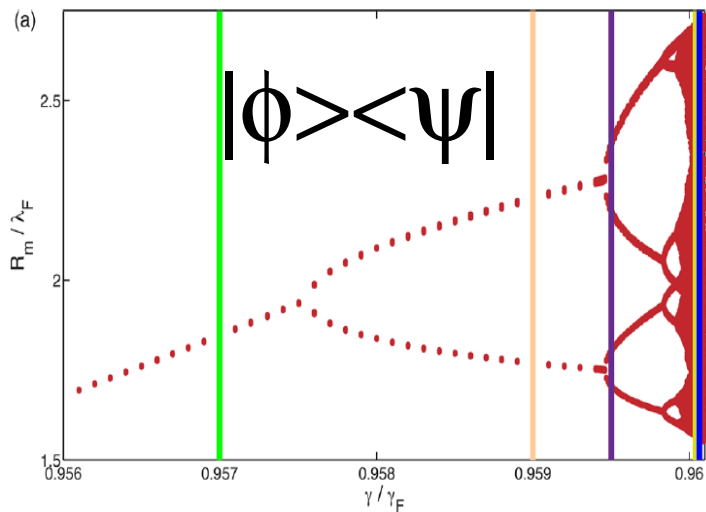
2 definitions of entropy and both have problems Boltzmannian has
Lokschmidt paradox/molecular chaos, Gibbsian has ergodicity
Khinchin's "failed" PhD to justify stat mech, Hilbert's 6th problem at
most stat mech approximate for some observables (indecomposability!)

They are different apart from certain limits (equilibrium, no-interaction)
Gibbs more general wrt systems but not defined away from equilibrium

Frequentist vs Bayesian One is more Boltzmannian, the other Gibbsian

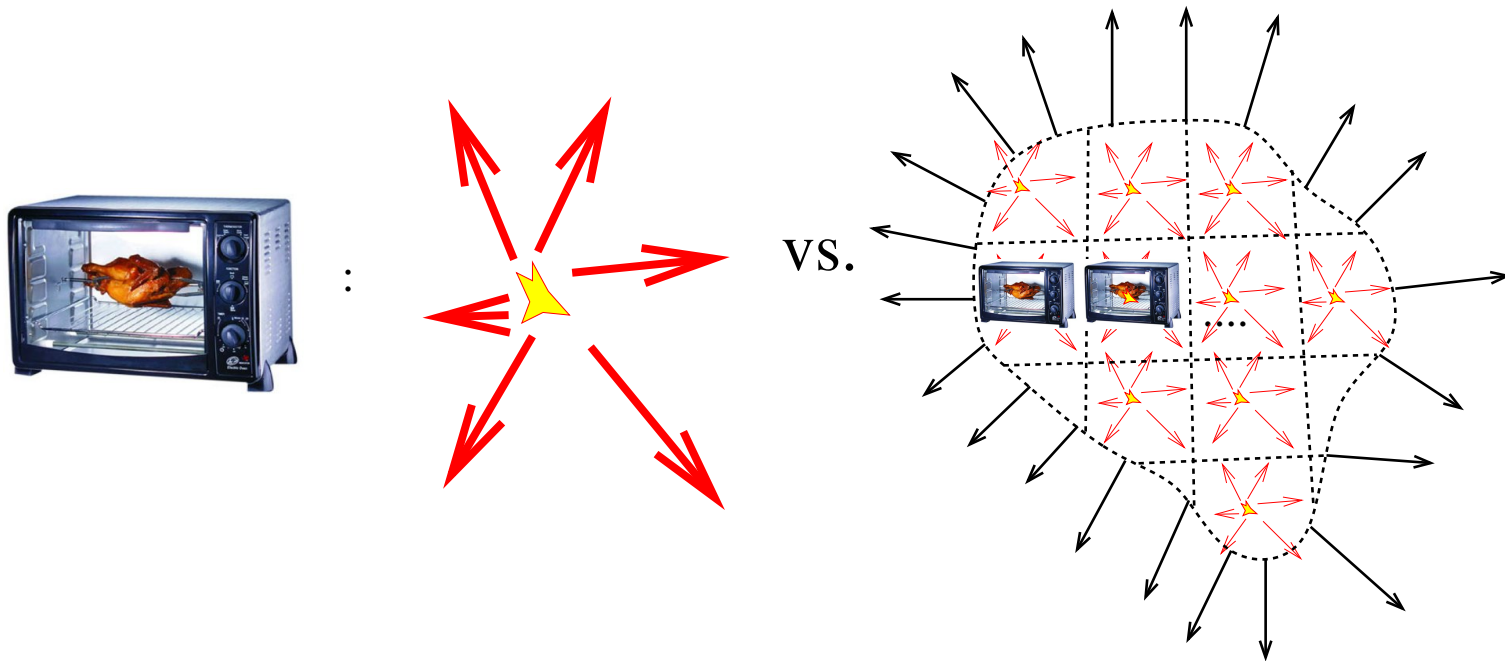
Relativity $T \rightarrow \beta_\mu \rightarrow$ time dependence! What's equilibrium?

QM to rescue? Berry/Bohigas/Eigenstate thermalization



$E_{n \gg 1}$ of quantum systems whose classical correspondent is chaotic have density matrices that look like pseudo-random. If off-diagonal elements oscillate fast or observables simple, indistinguishable from MCE!

Unlike Schwinger-Keldysh, Kubo, Boltzmann... fluctuations need not be small here **but**

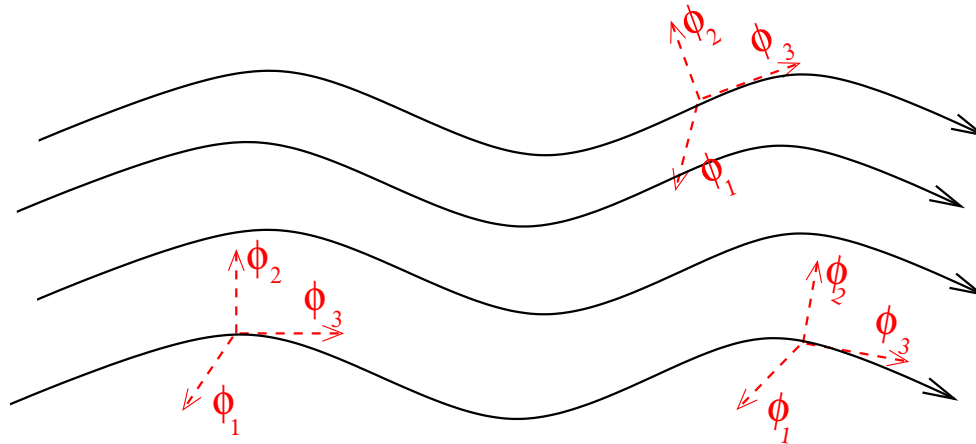


But need to coarse-grain, impose causality, and build hydro-like EFT out of this. could be very different from usual EFT expansion!

Thermodynamic limit relativistically ambiguous: “large volumes” can fluctuate into non-equilibrium sub-volumes that need “long times” to re-equilibrate . Grad limit/large N_c hides this!

Let's look at this ambiguity a bit deeper: Lagrangian and Eulerian hydrodynamics Hydro as fields: (Nicolis et al, 1011.6396 (JHEP))

Continuous mechanics (fluids, solids, jellies,...) is written in terms of 3-coordinates $\phi_I(x^\mu), I = 1...3$ of the position of a fluid cell originally at $\phi_I(t = 0, x^i), I = 1...3$. (Lagrangian hydro. NB: no conserved charges)



The system is a **Fluid** if its Lagrangian obeys some symmetries (Ideal hydrodynamics $\leftrightarrow$ Isotropy in comoving frame) Excitations (Sound waves, vortices etc) can be thought of as "Goldstone bosons"

Translation invariance at Lagrangian level $\leftrightarrow$ Lagrangian is a function of $B^{IJ} = \partial_\mu \phi^I \partial^\mu \phi^J$ Now we have a “continuous material”!

Homogeneity/Isotropy the Lagrangian is a function of $B = \det B^{IJ}, \text{diag} B^{IJ}$ fluid cell interior has no “preferred” direction $\Leftarrow SO(3)$

Invariance under Volume-preserving diffeomorphisms means the Lagrangian must be a function of B In all fluids a cell can be infinitesimally deformed

$$\rho = F(B) , \quad p = F(B) - 2F'(B)B , \quad u^\mu = \frac{1}{6\sqrt{B}} \epsilon^{\mu\alpha\beta\gamma} \epsilon_{IJK} \partial_\alpha \phi^I \partial_\beta \phi^J \partial_\gamma \phi^K$$

usual hydro energy-momentum tensor follows! $\sqrt{B}$ is identified with the entropy, $\sqrt{B} \frac{dF(B)}{dB}$ with microscopic temperature. u^μ fixed by $u^\mu \partial_\mu \phi^{\forall I} = 0$

Conserved charges (Dubovsky et al, 1107.0731(PRD))

Within Lagrangian field theory a scalar chemical potential is added by adding a $U(1)$ symmetry to system.

$$\phi_I \rightarrow \phi_I e^{i\alpha} \quad , \quad L(\phi_I, \alpha) = L(\phi_I, \alpha + y) \quad , \quad J^\mu = \frac{dL}{d\partial_\mu \alpha}$$

generally flow of b and of J not in same direction. Can impose a well-defined u^μ by adding chemical shift symmetry

$$L(\phi_I, \alpha) = L(\phi_I, \alpha + y(\phi_I)) \rightarrow L = L(b, y = u_\mu \partial^\mu \alpha)$$

A comparison with the usual thermodynamics gives us

$$\mu = y \quad , \quad n = dF/dy$$

obviously can generalize to more complicated groups

This looks a bit like GR and this is not a coincidence!

4D local Lorentz invariance becomes local $SO(3)$ invariance

Vierbein $g_{\mu\nu} = \eta^{\alpha\beta} e_\mu^\alpha e_\nu^\beta$ is $\frac{\partial x_I^{comoving}}{\partial x_\mu} = \partial_\mu \phi_I$ (with Gauge phase for μ)

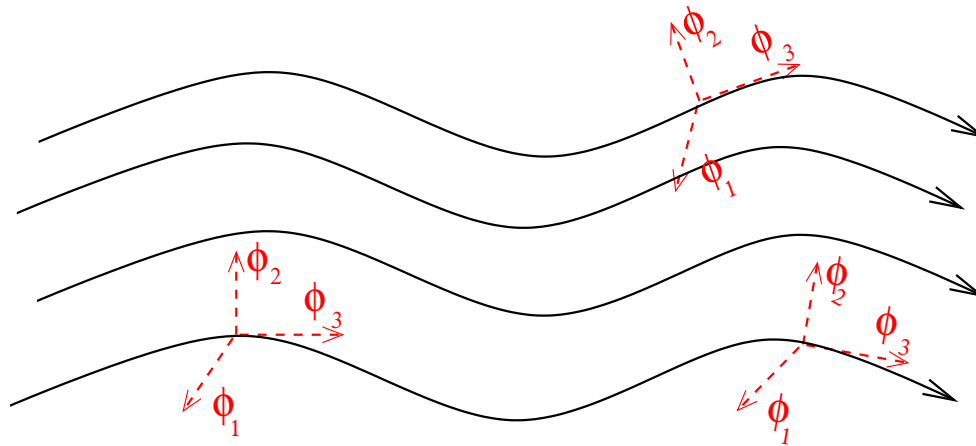
Entropy $\sim \sqrt{b}$, diffeomorphism invariant

Killing vector becomes u_μ

$\mathcal{L} \sim \sqrt{-g} (\Lambda + R + \dots)$ becomes $\mathcal{L} \sim F(B) \equiv f(\sqrt{-g})$ Just cosmological constant, expanding fluid $\equiv$ dS space

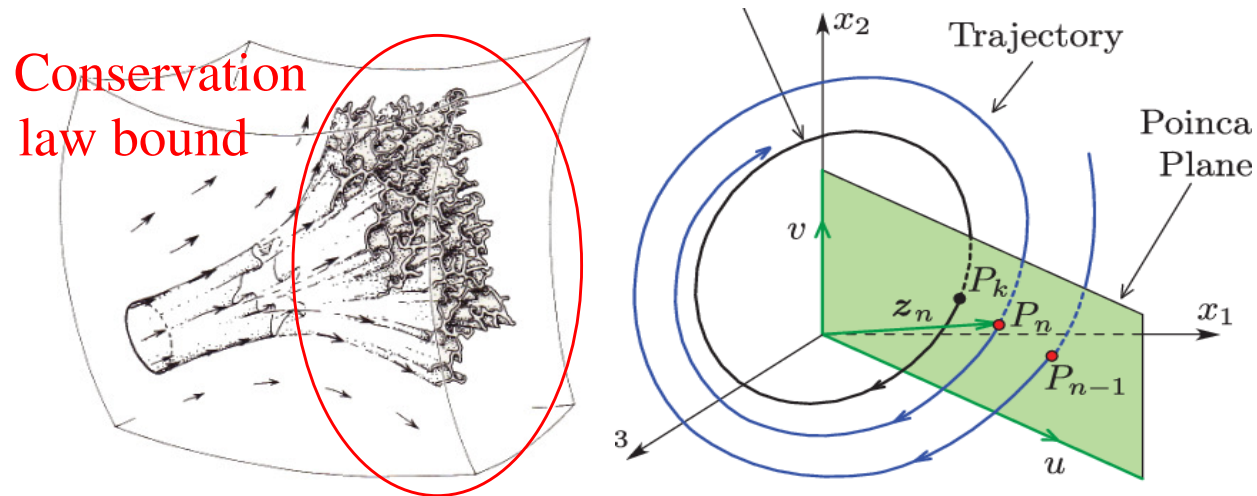
Very nice... but the ambiguities beyond ideal hydro generally break this .
Who cares? Should beyond idel hydrodynamics have this general covariance?

The poor people's quantum gravity: How can fluctuations and dissipation keep hydrodynamic's diffeomorphism invariance? Perhaps has a role to answer how come fluctuation/dissipation experimentally breaks down and fluids exist for 20 particles!



First step: Lagrangian hydrodynamics very elegant, but where is the connection to local thermalization? Statistical mechanics? Transport?
Hint from D.T.Son: it is the largest group of diffeomorphisms where time plays no role!

Where does statistical mechanics come from? Ergodicity



Classical evolution via Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H}{\partial x} \quad , \quad \dot{O} = \{O, H\}$$

"Chaos", conservation laws $\rightarrow$ phase space more "fractal", recurring

“After some time”, for any observable ergodic limit applies

$$\int_0^{(large) T} \dot{O}(p, q) dt = \int P(O(p, q)) dq dp$$

where $P(...)$ probability independent of time. This probability can only be given by conservation laws

$$P(O) = \frac{(\sum_i O_i) \delta^4 (\sum_i P_i^\mu - P^\mu) \delta (\sum_i Q_i - Q)}{N}, \quad N = \int P(O) dO = 1$$

this is the microcanonical ensemble. In thermodynamic limit

$$P(O) \rightarrow \delta(O - \langle O \rangle)$$

Hydrodynamics is “thermodynamics in every cell

$$\int_0^{(large) T} \dot{O}(p, q) dt \rightarrow \frac{\Delta\phi}{\Delta t}$$

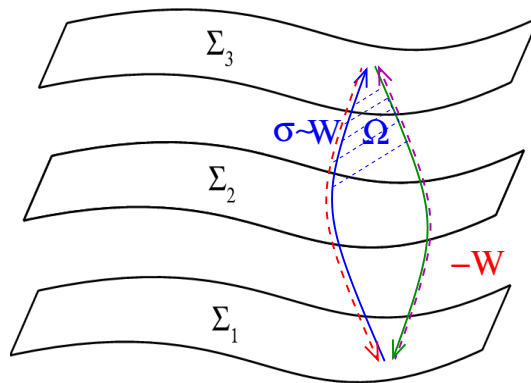
where ϕ is some local observable.

$$\left. \frac{\Delta\phi}{\Delta t} \right|_{t-t'=\Delta} \simeq \frac{1}{d\Omega(Q, E)} \times$$

$$\times \sum \delta_{P^\mu, P_{macro}^\mu(t)}^4 \delta_{Q, Q_{macro}(t)} \delta \left(\sum_j^\infty p_j^\mu - P^\mu \right) \delta \left(\sum_j^\infty Q_j - Q \right)$$

Problem: This is not relativistically covariant!

Solution: Foliation!



$$t \rightarrow \Sigma_0 \quad , \quad x_\mu \rightarrow \Sigma_\mu \quad , \quad \Delta \rightarrow \text{"smooth"} \quad \frac{\partial \Sigma_\mu}{\partial \Sigma_\nu}$$

Smooth: $R_{curvature}$ of metric change smaller than "cell size" (New l_{mfp})

$$\frac{\Delta \phi}{\Delta \Sigma_0} = \int P(\phi, \Sigma_\mu) d\Sigma_i \quad , \quad \Sigma_\mu \rightarrow \Sigma'_\mu \quad , \quad \frac{\Delta \phi}{\Delta \Sigma'_0} = \frac{\Delta \phi}{\Delta \Sigma_0}$$

What kind of effective lagrangian would enforce

$$\frac{\Delta\phi}{\Delta\Sigma_0} = \int P(\phi, \Sigma_\mu) d\Sigma_i \quad , \quad \frac{\Delta\phi}{\Delta\Sigma'_0} = \frac{\Delta\phi}{\Delta\Sigma_0}$$

with

$$P(\dots) \sim \delta\left(\sum_i P_i^\mu - P\right) \delta\left(\sum_i Q_i - Q\right)$$

Now Remember Noether's theorem!

$$p_\mu = \int d^3\Sigma^\nu T_{\mu\nu} \quad , \quad T_{\mu\nu} = \frac{\partial L}{\partial \partial^\mu \phi} \Delta_\nu \phi - g_{\mu\nu} L \quad , \quad \Delta_\nu \phi(x_\mu) = \phi(x_\mu + dx_\nu)$$

$$Q = \int d^3\Sigma^\nu j_\nu \quad , \quad j_\nu = \frac{\partial L}{\partial \partial^\mu \phi} \Delta_\psi \phi \quad , \quad \Delta_\psi \phi = |\phi(x)| e^{i(\psi(x) + \delta\psi(x))}$$

momentum generates spatial translations, conserved charges generate complex rotations!

Space-like foliations decompose

$$d\Sigma_\mu = \epsilon_{\mu\nu\alpha\beta} \frac{\partial \Sigma^\nu}{\partial \Phi_1} \frac{\partial \Sigma^\alpha}{\partial \Phi_2} \frac{\partial \Sigma^\beta}{\partial \Phi_3} d\Phi_1 d\Phi_2 d\Phi_3$$

where the determinant (needed for integrating out δ – *functions* is only in the volume part

$$\frac{\partial \Sigma'_\mu}{\partial \Sigma_\nu} = \Lambda^\nu_\mu \det \frac{d\Phi'_I}{d\Phi_J} \quad , \quad \det \Lambda^\nu_\mu = 1$$

Physically, Λ^ν_μ moves between the frame $d\Sigma^\mu_{rest} = d\Phi_1 d\Phi_2 d\Phi_3 (1, \vec{0})$

so lets try

$$\underbrace{L(\phi)}_{\text{microscopic DoFs}} \simeq L_{eff}(\Phi_{1,2,3})$$

with

$$\frac{\Delta\phi}{\Delta\Sigma_0} = \int P(\phi, \Sigma_\mu) d\Sigma_i \quad , \quad P(...) = \delta(...) \delta(...)$$

the general covariance requirement of $\frac{\Delta\phi}{\Delta\Sigma_0} = \frac{\Delta\phi}{\Delta\Sigma'_0}$ means the invariance of the RHS

$$\begin{aligned} & \frac{d\Omega(dP'_\mu, dQ', \Sigma'_0)}{d\Omega(dP_\mu, dQ, \Sigma_0)} = \\ &= \frac{d\Sigma'_0 \int da_\mu d\psi \delta^4 (d\Sigma^\nu a_\alpha \partial^\alpha (\delta^\mu_\nu L) - dP^\mu(\Sigma_0)) \delta (d\Sigma^\mu \psi \partial_\mu L - dQ(\Sigma_0))}{d\Sigma_0 \int da'_\mu d\psi' \delta^4 (d\Sigma'_\nu a'_\alpha \partial^\alpha (\delta^\mu_\nu L) - dP'_\mu(\Sigma'_0)) \delta (d\Sigma'_\mu \psi' \partial^\mu L - dQ'(\Sigma'_0))} \end{aligned}$$

It is then easy to see, via

$$\delta((f(x_i))) = \sum_i \frac{\delta(x_i - a_i)}{\underbrace{f'(x_i = a_i)}_{f(a_i)=0}} \quad , \quad \phi'_I = \frac{\partial_\alpha \Sigma'_I}{\partial^\alpha \Sigma^J} \Phi_J \quad , \quad \delta^4(\Sigma_\mu) = \det \left| \frac{\partial \Sigma^\mu}{\partial \Sigma^\nu} \right| \delta^4$$

that for general covariance to hold

$$L(\Phi_I, \psi) = L(\Phi'_I, \psi') \quad , \quad \det \frac{\partial \phi_I}{\partial \phi_J} = 1 \quad , \quad \psi' = \psi + f(\phi_I)$$

the symmetries of perfect fluid dynamics are equivalent to requiring
the ergodic hypothesis to hold for generally covariant causal spacetime
foliations!!!! Quantum: $\Delta t_{micro-sampling} \rightarrow \rho_{ij} e^{i\Delta t E_{ij}}$ and proof similar!

So we have a definition of ideal hydro ,independent of gradients and amenable to **large fluctuations** . **non-ideal hydro extension?**

Generating functionals , not constitutive relations Every cell corresponds to **covariant** $\ln \mathcal{Z}$, not **conserved current** metric $g_{\mu\nu} \leftrightarrow \partial \Sigma_\mu / \partial \Sigma^\nu$

Close to local equilibrium is **not** on gradient expansion but the approximate applicability of fluctuation-dissipation (**not the same!**)
Refoliations in $\Sigma_\mu \rightarrow$ Changes in $g_{\mu\nu} \leftrightarrow$ reshuffling in interpretation.

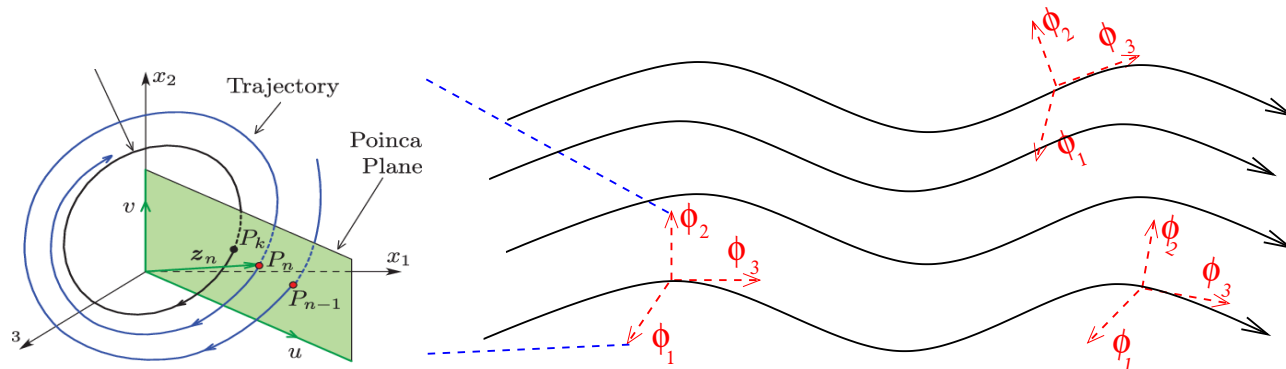
Manifestly in terms of probability distributions of observable quantities
 $T_{\mu\nu}, J_\mu, \Omega_{\alpha\mu\nu}$, **Cells** defined by full generating functionals,

As in Einstein's equations covariance $\rightarrow$ correct form of the EoM

Intrinsically incompatible with Boltzmann equation, needs functionals/autocorrelation

$\forall$ terms allowed terms compatible with symmetries i.e. **general covariance!**

How this helps with small systems Ergodicity/Poincare cycles meet relativity



Gibbs entropy+relativity : non-equilibrium $\rightarrow$ “phase loss” of Poincare cycles. one can see a slightly out of equilibrium cell either as a “mismatched u_μ ” (fluctuation) or as lack of equilibrium (dissipation). Microscopic momentum fluctuations and momentum dissipation locally indistinguishable!

Fluctuations reducible to $d\Sigma_\mu \rightarrow d\Sigma'_\mu$, shift in $u_\mu, \Pi_{\mu\nu}$ unphysical/“ghost”!

What is a gauge theory, exactly?

$$\mathcal{Z} = \int \mathcal{D}A^\mu \exp [S[F_{\mu\nu}]] \equiv \int \mathcal{D}A_1^\mu \mathcal{D}A_2^\mu \exp [S[A_1^\mu]]$$

$A_{1,2}^\mu$ can be separated since physics sensitive to derivatives of $\ln \mathcal{Z}$

$$\ln \mathcal{Z} = \Lambda + \ln \mathcal{Z}_G \quad , \quad \mathcal{Z}_G = \int \mathcal{D}A^\mu \delta(G(A^\mu)) \exp [S(A_\mu)]$$

Ghosts come from expanding $\delta(\dots)$ term. In KMS condition/**Zubarev**

$$Z = \int \mathcal{D}\phi e^{-\mathcal{F}}, \mathcal{F} \rightarrow d\Sigma_\nu \beta_\mu T^{\mu\nu} \quad , \quad Gauge : \mathcal{F}(\beta_\mu, \Sigma_\mu) \rightarrow \mathcal{F}(\beta'_\mu, \Sigma'_\mu)$$

Multiple $T_{\mu\nu}(\phi) \rightarrow$ **Gauge-like configuration** . Related to **Phase space fluctuations of ϕ** Small system fluctuations $\leftrightarrow$ UV fluctuations in QED?

A generally covariant local equilibrium theory:Ingredients

Partition functions approximately Gaussian the most “ad hoc” assumption, we do it because it works and hope Gaussians are universal enough **problematic for critical points**

The Gravitational ward identity For Gaussians, and only for them it both enforces general covariance and determines dynamics

Fluctuation-dissipation needed for 2nd law
And needs to be generally covariant for consistency!

The gravitational ward identity (Deser, Boulware, JMP **8** (1967), 1468)

$$\nabla \mathcal{W} = 0 \quad , \quad \mathcal{W} = G^{\mu\nu, \alpha\beta} (\Sigma_\mu, \Sigma'_\nu) - \frac{1}{\sqrt{g}} \delta (\Sigma' - \Sigma) \times$$

$$\times \left(g^{\beta\mu} \left\langle \hat{T}^{\alpha\nu} (x') \right\rangle_\Sigma + g^{\beta\nu} \left\langle \hat{T}^{\alpha\mu} (x') \right\rangle_\Sigma - g^{\beta\alpha} \left\langle \hat{T}^{\mu\nu} (x') \right\rangle_\Sigma \right)$$

$G^{\mu\nu, \alpha\beta}$ is the propagator for energy-momentum tensor (fluctuation!)

Kubo formulae neglect contact terms by imposing thermostatic background

Fancy name but consequence of energy-momentum Noether current

$$\partial_\mu T^{\mu\nu} + \Gamma_{\nu\alpha\beta} T^{\alpha\beta} = 0 \quad , \quad \langle T_{\mu\nu}^n \rangle = \frac{\delta^n}{\sqrt{-g} \delta g^{\mu\nu(n)}} \ln \mathcal{Z}$$

Note: deterministic hydro gradient expansion and linearized fluctuations **inherently break this!** Ward identity specifies dynamics only for Gaussian

Cumulant expansion: Partition function is Gaussian! $\ln \mathcal{Z} \simeq \ln \mathcal{Z}|_0 -$

$$- \frac{\partial^2 \ln \mathcal{Z}}{\partial \beta_\mu \partial \beta_\nu} \Big|_0 \ln \prod_{\Sigma(x)} \exp \left[-\frac{1}{2} \langle \Delta T_{\mu\nu}(\Sigma(x')) \rangle C^{\mu\nu\alpha\beta}(\Sigma(x), \Sigma(x')) \langle \Delta T_{\alpha\beta}(\Sigma(x)) \rangle \right]$$

Gaussian $\ln \mathcal{Z}$ is Solvable, function of diagonalized $\langle T_{\mu\nu} \rangle$ (e,p) and $C'^{\mu\nu\alpha\beta}$ in frame that diagonalizes $\langle T_{\mu\nu} \rangle$

$$\ln \mathcal{Z}(\Sigma_0) \simeq \left(\sqrt{\prod_i \mu_i^2} \sqrt{C'_{\alpha\beta}} \right)^{-1}$$

where

$$C'_{\alpha\beta\iota\xi} = \Lambda_{\mu\nu} \Lambda_{\alpha\beta} \Lambda_{\delta\iota} \Lambda_{\zeta\xi} C^{\mu\alpha\delta\zeta} \quad , \quad \Lambda^{\mu\nu} \Lambda^{\alpha\beta} \langle T_{\mu\beta} \rangle = \text{Diag}(\mu_1, \mu_2, \mu_3, \mu_4)$$

If $C_{\alpha\beta\gamma\nu}$ evolved via Ward identity from initial conditions **manifestly diffeoinvariant**

$$Q^{\alpha\beta} = \int d\Sigma_\mu \left[\nabla_\nu C^{\mu\nu\alpha\beta} + \frac{1}{\sqrt{-g}} \sum_i \frac{1}{\lambda_i^2 \sqrt{C'_{\zeta\delta}}} \nabla^\mu C^{\alpha\beta}_{\mu\nu} \right]$$

Physically, “the price” for general covariance is that the propagator is not a function of the Lagrangian, but **evolved deterministically from initial conditions in an ensemble of field configurations** **This propagator can then be used to propagate the ensemble, and vice-versa!**

Lesson to make other stochastic theories generally covariant (Gravity)?

Temperature obtainable from covariantization of Maxwell relations

$$\langle E^2 \rangle - \langle E \rangle^2 \equiv C_V T \Rightarrow \left(C'_{\alpha\beta} \right)^{3/2} = \frac{\partial \beta_\mu \partial \beta_\nu}{\partial \Sigma_\mu \partial \Sigma_\nu}$$

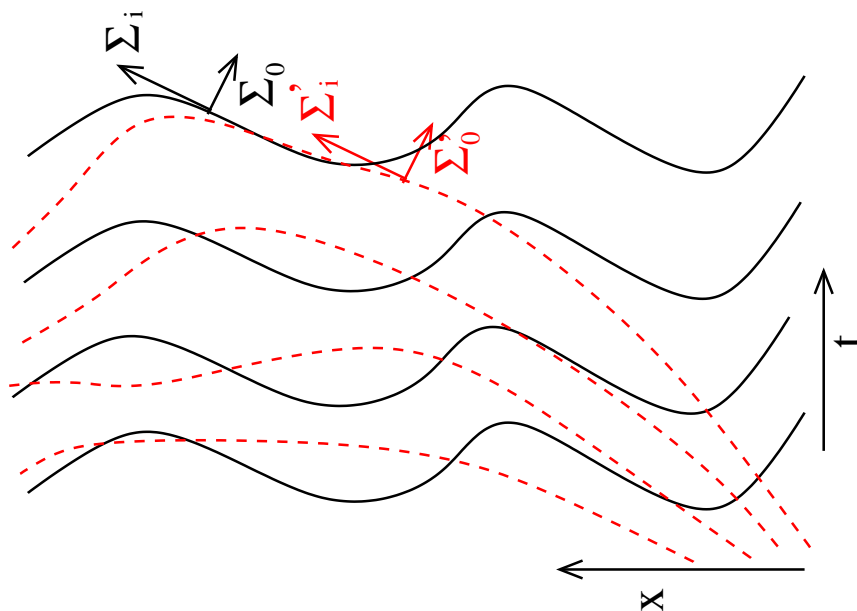
$\tilde{C}'_{\alpha\beta\mu\nu}(k)$ contains EoS and transport coefficients information

$$\eta \sim k^{-1} \lim_{k \rightarrow 0} \text{Im} \tilde{C}'_{xyxy} \quad , \quad c_s^2 \sim k^{-1} \lim_{k \rightarrow 0} \text{Im} \tilde{C}'_{xxxx}$$

$$\tau_\pi \sim k^{-2} \lim_{k \rightarrow 0} \text{Im} \frac{d}{dk} \tilde{C}'_{xyxy} \quad C_V \sim \max_k \text{Re} \tilde{C}'_{xxxx} \quad , \quad \dots$$

But we should underline that the full $\tilde{C}'_{\alpha\beta\mu\nu}(k)$ is a field that evolves from the initial conditions of an ensemble of configurations of the $T_{\alpha\beta}$ field, not a functional of the lagrangian alone

General covariance and linear response



Different foliations are locally causal, but disagree on causality at distant regions. Linear response, Schwinger Keldysh,... fluctuation (equal time correlator) and dissipation (forward correlator) relation complicated but you “know the background” . general covariance emerges in limit you do not!

fluctuation-dissipation relation From linear response the Gaussian propagator earlier,

$$T_{\mu\nu}(\Sigma) = \int e^{\epsilon\Sigma_0} G^{\mu\nu,\alpha\beta}(\Sigma'_0 - \Sigma_0) \delta g_{\alpha\beta}(\Sigma'_0) d\Sigma_0$$

we can find the time-ordered one

$$\langle \hat{T}^{\alpha\beta} \rangle_{\Sigma+d\Sigma} = i \int \frac{1}{\sqrt{g}} \mathcal{G}^{\mu\nu,\alpha\beta}(\Sigma'_0 - \Sigma_0) \langle T \rangle_{\mu\nu}(\tau, x) d\Sigma_0$$

$$\tilde{\mathcal{G}}^{\mu\nu\alpha\beta} = \frac{1}{2i} \left(\frac{\tilde{G}_T^{\alpha\beta\mu\nu}(\Sigma_0, k)}{\tilde{G}_T^{\alpha\beta\mu\nu}(-i\epsilon\Sigma_0, k)} - 1 \right)$$

These are Standard fluctuation-dissipation techniques (Forster, Kadanoff, ...) but are they generally covariant

But Can it be Generally covariant? Seemingly no! in $d\omega$ and because

$$dV dt \rightarrow \sqrt{|\det [-g_{\mu\nu}]|} d^4x \quad , \quad \prod_i \lambda_i \mu_i C'_{\mu\nu}{}^{\mu\nu} \sim Volume$$

restored if volume preserving diffeomorphisms invariance

$$\Sigma_\mu = (t(x), \vec{x}) = (t(\phi_I), \phi_{I=1,2,3}) \quad , \quad \phi_I \rightarrow \phi'_I(\phi_J) \quad , \quad \det_{IJ} \partial\phi_I / \partial\phi'_J = 1$$

hydro EFTs based on Schwinger-Keldysh break this symmetry beyond ideal limit (Haehl et al,1502.00636,also Grozdanov,GT+Montenegro) , which shows role of general covariance . Physically Local equilibrium $\rightarrow$ dissipative evolution locally indistinguishable from an isentropic fluctuation. so symmetries of ideal hydro should remain! flow in ideal limit is defined as a Killing vector, $u^\mu \partial_\mu \phi_I = 0$ Physically, once fluctuations are taken into account in a generally covariant way, u_μ deviation from this is locally indistinguishable from a fluctuation, encoded in the structure of $C_{\mu\nu\alpha\beta}$.

The dynamics (Lattice version)

Start with a lattice with an ensemble $\{T_{\mu\nu}(x_i, t)\}$ at each point of it

Construct $\langle T_{\mu\nu}(x_i, t) \rangle$ and $C_{\mu\nu\alpha\beta}(x_i, x_j, t, t)$ from it

Use General covariance and $C_{\mu\nu\alpha\beta}(x_i, x_{i+1}, t)$
to construct $\mathcal{G}_{\alpha\beta\mu\nu}(x_i, x_{i+1}, t)$

Use linear response on each $\{T_{\mu\nu}(x_i, t)\}$ to get $\{T_{\mu\nu}(x_i, t + \Delta t)\}$

Calculate $\langle T_{\mu\nu}(x_i, t + \Delta t) \rangle$ Use the Ward
identity to get $C_{\mu\nu\alpha\beta}(x_i, x_{i+1}, t + \Delta t)$

The mere fact that thermodynamic quantities can be described via a $\ln \mathcal{Z}$ gives rise to the Gibbs-Duhem relation

$$s = T \ln \mathcal{Z} = P + e - \mu n$$

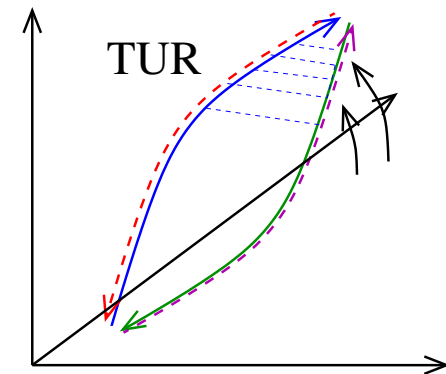
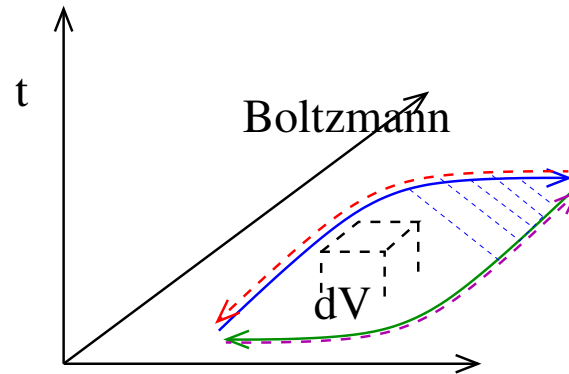
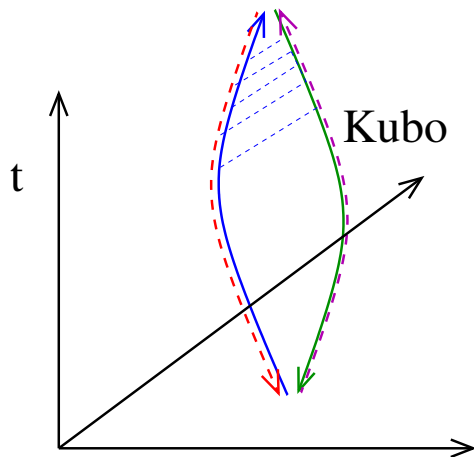
Enforce invariance under Σ_μ refoliations, a scalar $\ln \mathcal{Z}$

$$-\Delta \ln \mathcal{Z} = -\beta_\nu J^\nu \Delta \mu + P^i \Delta \beta_i - \Delta \Sigma^0 \beta_0 \int_0^{P^0} c_s^2(e) de \quad , \quad P_{\alpha=0, i=1,2,3} \equiv T_{\alpha\beta} d\Sigma^\beta$$

Crooks theorem becomes

$$\frac{\mathcal{P} \left\{ P_\mu|_\tau \rightarrow P_\mu|_{\tau+\Delta\tau} \right\}}{\mathcal{P} \left\{ P_\mu|_{\tau+\Delta\tau} \rightarrow P_\mu|_\tau \right\}} \sim \exp[\ln \mathcal{Z}|_{\tau+\Delta\tau} - \ln \mathcal{Z}|_\tau] \quad , \quad \Delta\tau = \beta_\mu \frac{\Delta^3 \Sigma^\mu}{\Delta^3 \phi_{i=1,2,3}}$$

Zubarev statistical operator + Crooks theorem (Work $\sim \int T_{\mu\nu} d^3 \Sigma^\mu \beta^\mu d\tau$)



Crooks fluctuation theorem Relates fluctuations, entropy, work

$$P(W)/P(-W) = \exp [\Delta S] \quad , \quad W = T_{\mu\nu} d\Sigma^\mu \beta^\nu$$

reproduces Boltzmann entropy and Kubo for right contours and...

ideal hydro emerges for small fluctuations and vanishing viscosity!

$$\lim_{\Delta S \rightarrow \infty} \frac{P(W)}{P(-W)} = e^{\Delta S} \equiv \delta(d\Sigma_\mu(su^\mu)) = 0 \Rightarrow n^\mu \partial_\mu(su^\mu) = 0$$

Singular limit but regularizeable Fluctuation constraint becomes singular when general covariance determines EoM!

Characteristic volume of volume preserving diffeomorphisms $\rightarrow 0$

Viscous corrections break volume-preserving diffeos because fluctuation-dissipation violated. Any structure of a distribution invisible when it collapses to a δ -function. Fluctuation and dissipation appear together in a way satisfying the Ward identity!

The dynamics (Lattice Crooks version)

Start with a lattice with an ensemble $\{T_{\mu\nu}(x_i, t)\}$ at each point of it

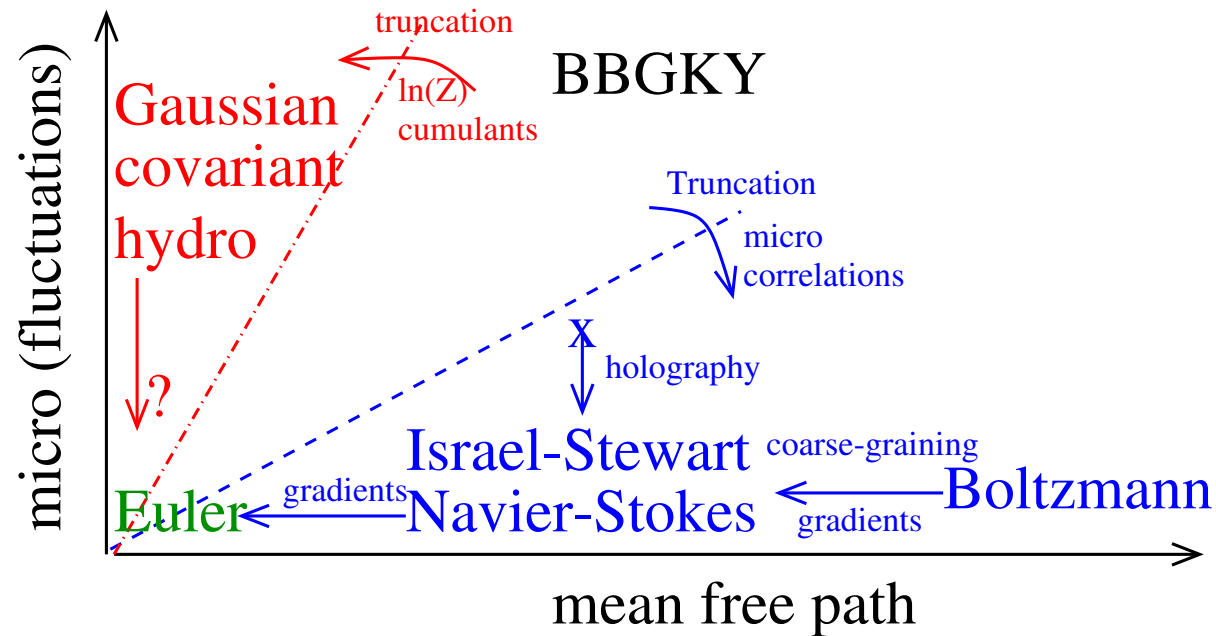
Construct $\langle T_{\mu\nu}(x_i, t) \rangle$ and $C_{\mu\nu\alpha\beta}(x_i, x_j, t, t)$ from it

Use General covariance via $d\Sigma_\mu|_{timelike}$ to
 $d\Sigma_\mu|_{spacelike}$ to find $C_{\mu\nu\alpha\beta}(x_i, x_{i+1}, t + \Delta t, t + \Delta t)$

Use the Crooks fluctuation theorem and
acceptance/rejection to get $\{T_{\mu\nu}(x_i, t + \Delta t)\}$

Calculate $\langle T_{\mu\nu}(x_i, t + \Delta t) \rangle$ Use the Ward
identity to get $C_{\mu\nu\alpha\beta}(x_i, x_{i+1}, t + \Delta t)$

Those scales again: $\underbrace{l_{micro}}_{\sim s^{-1/3}, n^{-1/3}} \ll \underbrace{l_{mfp}}_{\sim \eta/(sT)} \ll L_{macro}$



Preponderance of “chaotic” regime could mean general covariant hydro is both more likely and “closer in dynamics” to Euler!

When is the system is a good fluid? Usual approach, Kn , we propose ξ

$$Kn \equiv \min_{\beta} \frac{(T^{\mu\nu} - T_0^{\mu\nu}) \partial_{\mu} \beta_{\mu}}{T_{\alpha}^{\alpha}} \sim \frac{\eta}{(e + p)R} \sim \frac{\tau_{\Pi}}{R}, \xi \equiv \frac{\min_{\beta} [\langle T^{\mu\nu} - T_0^{\mu\nu} \rangle \partial_{\mu} \beta_{\mu}]}{\sqrt{C'_{\alpha\beta}}}$$

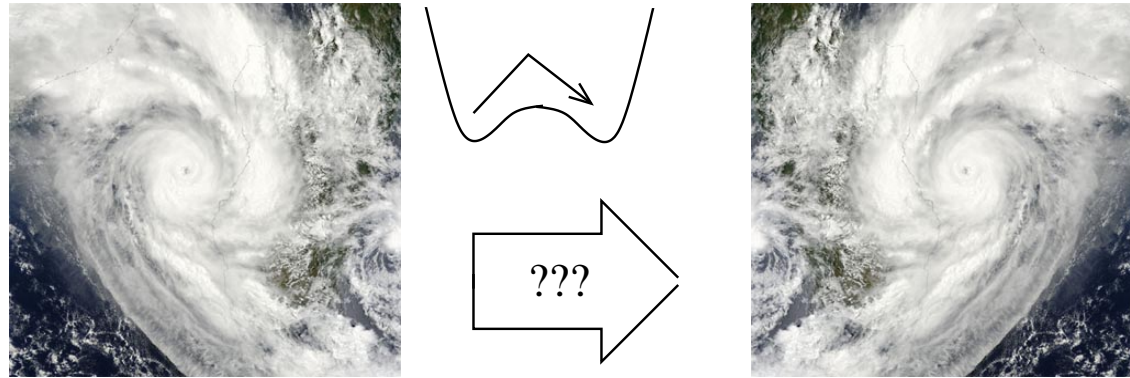
Flow ambiguity at most by the possibility to "choose" the vector β

fluctuations some fluctuation-dissipation relation $Kn \sim N_{dof}^{-1}$

Phenomenology breaks this

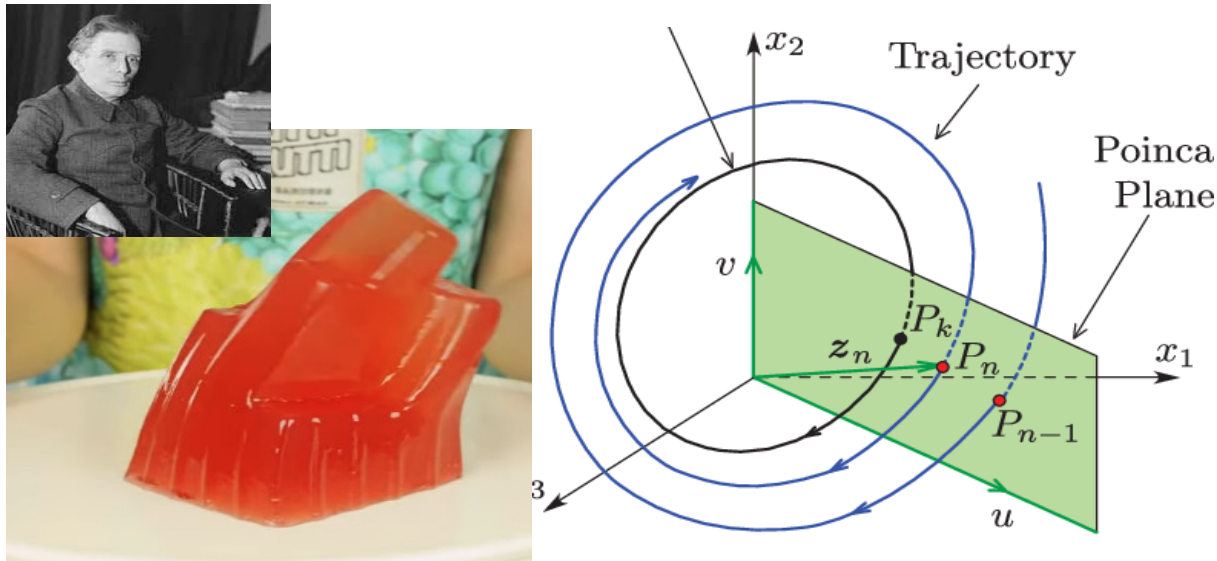
ξ qualitatively depends in a radically different way from on N_{dof} : As $N_{dof} \rightarrow \infty$ Gibbs-Duhem $\rightarrow \xi \sim Kn$, but away from this limit Kn is bound to increase with the number of DoFs while ξ could well decrease.
For small strongly coupled systems $\xi \ll 1$?

Hydrodynamics is universal I: local rather than global equilibrium!



Global equilibrium , defined as $\text{Max} [\langle \ln \hat{\rho} \rangle]_{\beta_{\mu}, \mu, \dots}$ in presence of forces (rotations, fields, acceleration) ill defined if $\nabla \delta_{\mu} \simeq 1/R, 1/T$ since **hydrodynamic turbulence, statistical fluctuations talk** (“unstable” equilibrium is not in equilibrium!). **local equilibrium** well-defined!, solid basis of an EFT. This ambiguity, reflecting the tension of **thermodynamic limit** and **relativity** , is due to entropy in Global equilibrium being Boltzmannian (“micro” Dofs) and not Gibbsian (covariantly “coarse-grained” Dofs, fluctuation-generated soundwaves,...)

Hydrodynamics is universal II: Solids, jellies etc and local equilibrium



Khinchin showed ergodicity impossible unless phase space indecomposable, something problematic for systems with broken volume preserving diffeos. Anything except fluids has a global equilibrium but no EFT built around local equilibrium stable against local fluctuations. **A complicated way of saying non-fluids are brittle (long correlations determine dynamics)**

Hydrodynamics is universal III: Hydrodynamic stability and causality

Stability? It's all about those scales

$$l_{micro} \ll l_{mfp} \ll L_{macro}$$

When statistical mechanics applies, one expects dynamics to be absolutely chaotic near the *first scale, where Poincare cycles occur* and partially chaotic in the middle of the chain of inequalities *where turbulence occurs*

Local causality? criterion must be changed to

$$\int dx e^{ikx} \langle u_\mu(x) u_\nu(x') \rangle \rightarrow \int dx e^{ikx} T [T_{\mu\nu}(x), T_{\alpha\beta}(x')]$$

fluctuations or signals between unobservables don't have to be causal!

Onto spin hydrodynamics?

STAR
collaboration
1701.06657

NATURE
August 2017

Polarization by vorticity
in heavy ion collisions



Could give new talk about this, but will mention hydro with spin not developed and a lot of conceptual debates. **does hydrodynamics with spin (and fluctuations) depend on the pseudo-gauge?** **entropy current** seems to!

What is a pseudogauge?

Mathematically (S. Jeon, 2310.11269) Let $\Phi^{\alpha\beta\gamma}$ be fully antisymmetric

$$T_{\mu\nu} \rightarrow T'_{\mu\nu} + \frac{1}{2} \partial_\lambda (\Phi^{\lambda,\mu\nu} + \Phi^{\mu,\nu\lambda} + \Phi^{\nu,\mu\lambda}) \quad , \quad \partial^\mu T_{\mu\nu} = \partial^\mu T'_{\mu\nu} = 0$$

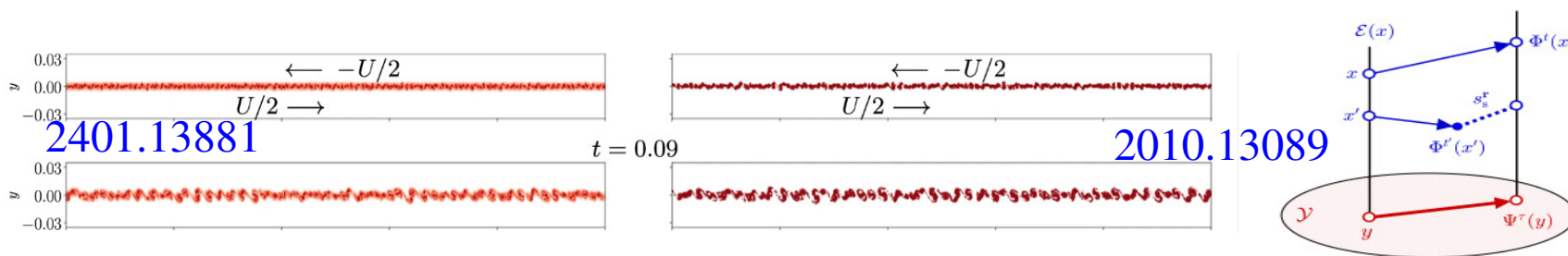
Can move around spin and angular momentum, shows the ambiguity of localized currents

Physically (T. Brauner, 1910.12224) a microscopic field redefinition with a non-inertial transformation preserving the action

$$x^\mu \rightarrow x^\mu + \epsilon \zeta^\mu(x) \quad , \quad \psi_a \rightarrow \psi_a + \epsilon \psi'_a \quad , \quad \mathcal{S} \rightarrow \mathcal{S}$$

So in current approach $\ln \mathcal{Z}$ derivatives (dynamics and observables) exactly pseudo-gauge invariant! Work in progress, need Ward identity with torsion

What happens in non relativistic limit?



- No unique Lorentz $\rightarrow$ Galileo expansion, need extra assumptions. I think
 - $c_s \rightarrow \infty \Rightarrow$ incompressible matter
 - This way Local hydro Lorentz invariance becomes the Galileo group
 - Spontaneous stochasticity conjecture [2401.13881](#)
 - Hidden spatiotemporal symmetries in turbulence [2010.13089](#)
- But regarding ergodic hypotheses, what is the Stat.mech of incompressible matter? Probably singular

Fluctuations in non-ideal hydrodynamics not well understood

Intimately related to entropy current, double counting of DoFs
Could alter fluctuation-dissipation expectation, "fluctuations help dissipate", in analogy to Gauge theory

Approximate local equilibrium not understood in Gibbsian picture
Our proposal: applicability of fluctuation-dissipation

Need a covariant description purely in terms of observable quantities
Ergodicity works in ideal hydro, Crooks theorem/K-K beyond it?

Could be relevant for hydro in small systems

A non-relativistic limit? (Brazil nut effect) all depends on time dilation, so a bit at a loss!

SPARE SLIDES

Fluctuations... what is fluctuating?

One can always decompose $T_{\mu\nu}$ into “near-equilibrium” e, p, n, u_μ and non-equilibrium “rest” $\Pi_{\mu\nu}, J_\mu$, its an algebraic operation, provided you have a matching condition eg

$$T_{\mu\nu}u^\mu = eu_\nu \quad , \quad T_{\mu\nu} = (e + p(e))u_\mu u_\nu + p(e)g_{\mu\nu} + \Pi_{\mu\nu}$$

At deterministic level this is good but consider an ensemble $\hat{T}_{\mu\nu}$...

$T_{\mu\nu} \rightarrow \hat{T}_{\mu\nu}$ so $e, p, n \rightarrow \hat{e}, \hat{p}, \hat{n}$, ensemble elements which fluctuate event-by-event. $u_\mu \rightarrow \beta_\mu$, Lagrange multiplier for momentum. Mixing these mixes apples and Legendre-transformed apples, lose all connection to stat-mech!

It would mean transversality condition $u_\mu \Pi^{\mu\nu} = 0$ applies event-by-event rather than on average, as expected from statistical mechanics .

Anisotropy, transport and statistical mechanics Anisotropic hydrodynamics justified within transport via improved relaxation time

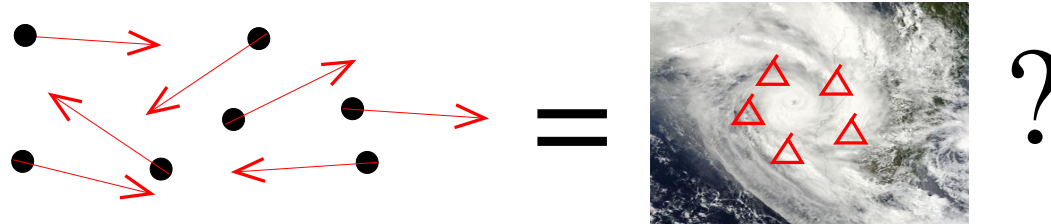
$$f(x, p) = f_{eq} (1 + \phi(x, p)) \rightarrow f_{eq} (1 + \phi(x, p) + a_\mu(x) p^\mu)$$

Problem: Boltzmann is an approximation where $f(x, p)$ represents **an infinity of particles** . Fundamentally, hydrodynamics comes from Kubo

$$\eta = \lim_{k \rightarrow 0} \frac{1}{k} \text{Im} \int dx \left\langle \hat{T}_{xy}(x) \hat{T}_{xy}(y) \right\rangle \exp [ik(x - y)]$$

Usually semiclassical approximation yields Boltzmann equation than relaxation time, which guarantees the Kinchin condition to be fulfilled. Above demonstration reliable only in that limit

The basic problem with $f(x,p)$



Let's solve the simplest transport equation possible: Free particles

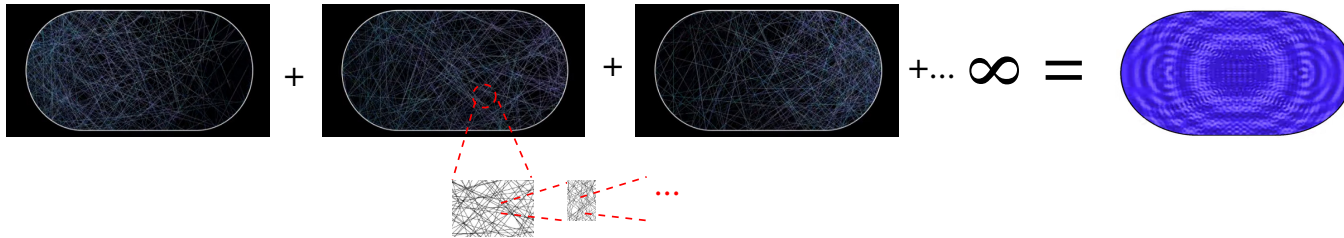
$$\frac{p^\mu}{m} \partial_\mu f(x, p) = 0 \rightarrow f(x, p) = f\left(x_0 + \frac{p}{m}t, p\right)$$

obvious solution is just to propagate

What is weird is that "hydro-like" solution possible too (eg vortices)!

$$f(x, p) \sim \exp[-\beta_\mu p^\mu] \quad , \quad \partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0$$

But obviously unphysical, no force! What's up?



This paradox is resolved by remembering that $f(x, p)$ is defined in an ensemble average limit where the number of particles is not just “large” but **uncountable** . **curvature from continuity!**

BUT this suggests Boltzmann equation disconnected from $N_{dof} \leq \infty$!

In Anisotropic hydro β_μ not Killing vector . So no reason to assume ensemble average/thermal fluctuations sampled fairly close to equilibrium! Boltzmann equilibrium and Gibbs-type thermal equilibrium could be very different. **lets work with the latter**

Vlasov and Boltzmann in a classical world

Villani , <https://www.youtube.com/watch?v=ZRPT1Hzze44>

Vlasov equation contains all classical correlations, instability-ridden, “filaments”, cascade in scales.

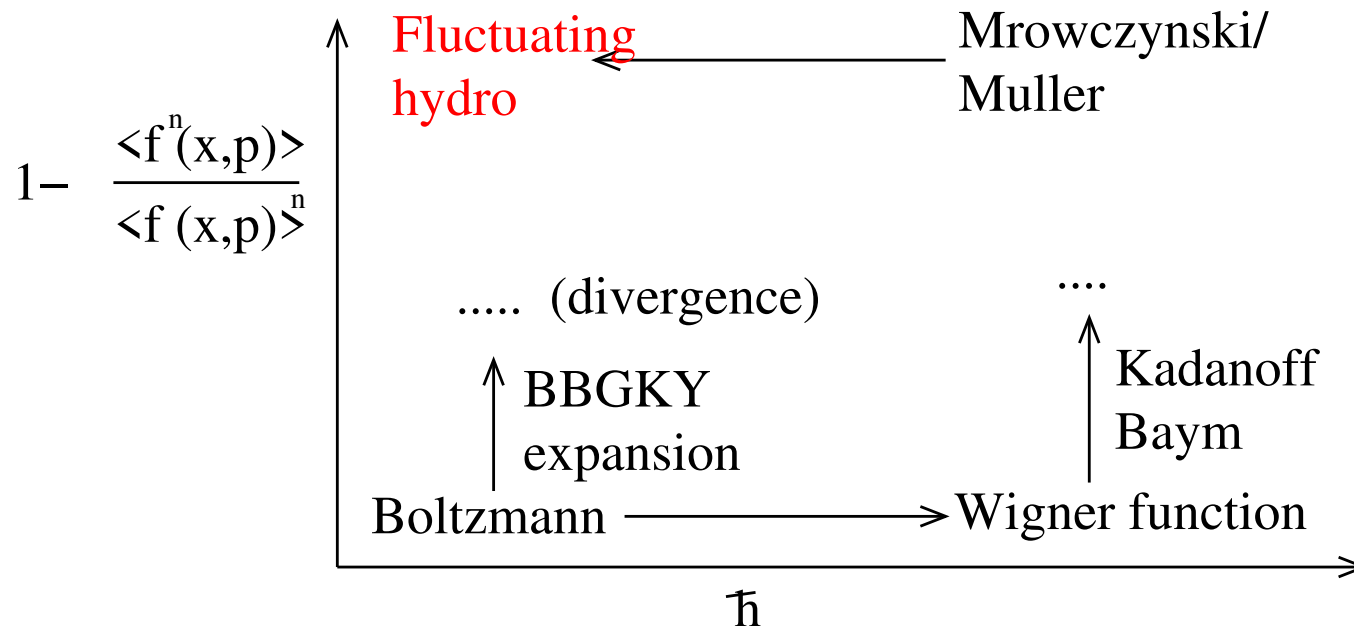
$N_{DOF} \rightarrow \infty$ invalidates KAM theorem stability

Boltzmann equation “Semi-Classical UV-completion” of Vlasov equation, first term in BBGK hierarchy, written in terms of Wigner functions.

Infinitely unstable jerks on infinitely small scales Random scattering

But if number of particles $N \ll \infty$ Correlations important! .

Boltzmann equation, BBGKY and limits



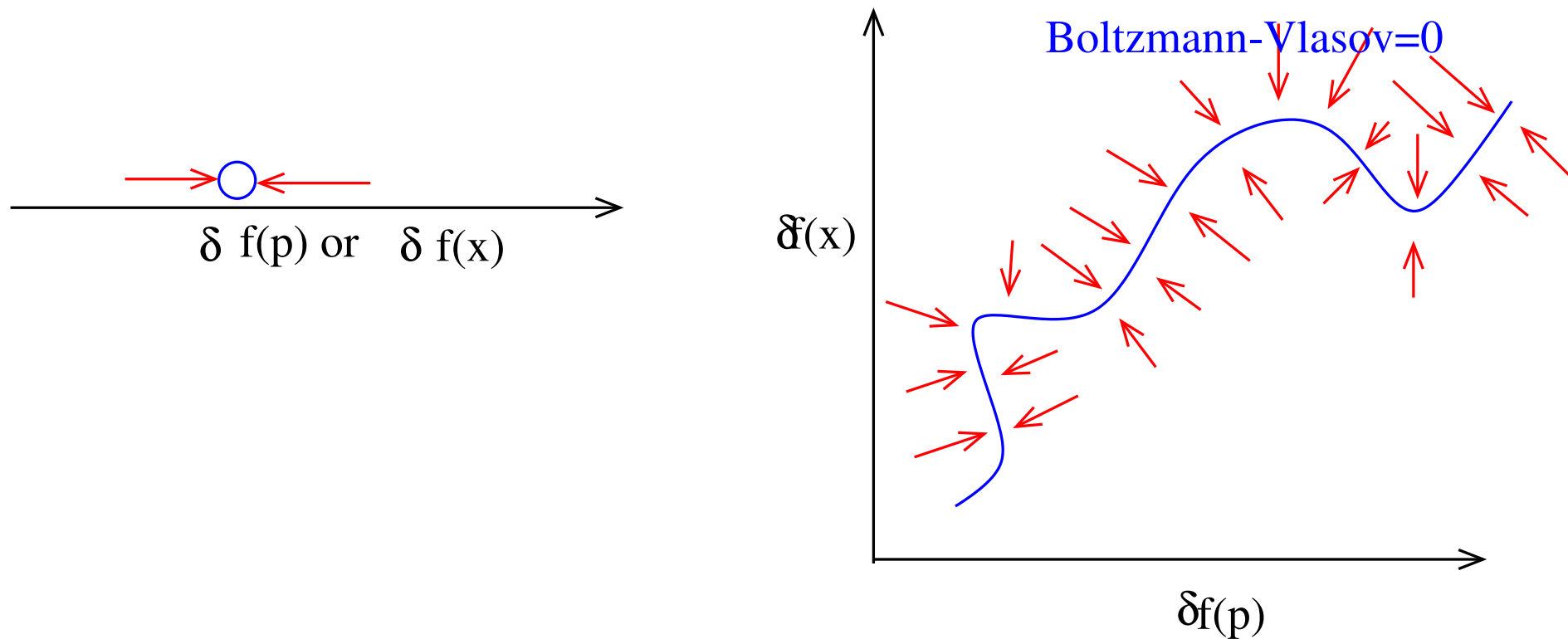
Boltzmann equation emerges as a double limit from **microscopic correlations**, $\hbar \rightarrow 0$. Relaxing the latter limit would destroy statistical independence **CHSH relations**, so probably not relevant (phases "chaotic"). But fluctuating hydro "non-perturbative" in correlations

Finite number of particles: $f(x, p)$ not a function but a functional
 $(\mathcal{F}(f(x, p)) \xrightarrow{\text{Boltzmann}} \delta(f' - f(x, p)))$, incorporating continuum of
 functions and all correlations. Perhaps solvable!

$$\frac{p^\mu}{\Lambda} \frac{\partial}{\partial x^\mu} f(x, p) = \left\langle \underbrace{\hat{C}[\tilde{W}(\tilde{f}_1, \tilde{f}_2)] - g \frac{p^\mu}{\Lambda} \hat{F}^{\mu\nu}[\tilde{f}_1, \tilde{f}_2] \frac{\delta}{\delta \tilde{f}_{1,2}} \tilde{W}(\tilde{f}_1, \tilde{f}_2)}_{\text{How many } A-B=0?} \right\rangle$$

Wigner functional to $\mathcal{O}(\hbar^0)$. What is the effect? If only Boltzmann term
 not much!

If Both Vlasov and Boltzmann terms, redundancy-ridden!



One can deform $f(x, p)$ by $\delta f(x)$ or $\delta f(p)$ so that $\hat{C} - \hat{W}$ cancels. In ensemble average deformation makes no sense, but away from it it does!

Discretize $x, p \rightarrow$ random matrix problem!

$$\dot{f}_{ij} - \left[\frac{\vec{p}_k}{\Lambda} \cdot \Delta_k \right] f_{ij} = \langle \hat{\Omega} \rangle$$

$$\hat{\Omega} \sim d[f'_{i_1 j_1}] \left[\mathcal{W}_{i_1 j_1 i j} \left(\mathcal{C}_{j j_1} (f_{ij} f'_{i_1 j_1} - f_{i j_1} f'_{i_1 j}) - \mathcal{V}_{ii_1}^\mu f_{ij} f'_{i_1 j_1} \frac{\Delta f_{ij}}{\Delta p^\mu} \right) \right]$$

- Theorems of random matrix theory can be used to prove limit very different from RTA!
- can be tested numerically with a **lattice Boltzmann** algorithm
- connects to Zubarev Gibbs-Duhem relation

$$\ln \mathcal{Z} = \ln \left[\prod_{i=1}^N \exp \left(\Delta^3 \Sigma_\mu (\beta_\nu T^{\mu\nu} - \mu_i J^\mu) \right) \right]$$